

# Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$

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## Abstract

We study the following singularly perturbed nonlocal Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-2} \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \quad \text{in } \mathbb{R}^2,$$

where  $V(x)$  is a continuous real function on  $\mathbb{R}^2$ ,  $F(s)$  is the primitive of  $f(s)$ ,  $0 < \mu < 2$  and  $\varepsilon$  is a positive parameter. Assuming that the nonlinearity  $f(s)$  has critical exponential growth in the sense of Trudinger-Moser, we establish the existence and concentration of solutions by variational methods.

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# 1 Introduction and main results

The nonlocal elliptic equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-N} \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \quad \text{in } \mathbb{R}^N, \quad (SNS)$$

the so-called Choquard equation when  $N = 3$ , appears in the theory of Bose-Einstein condensation and is used to describe the finite-range many-body interactions between particles. Here  $V(x)$  is the external potential,  $F(s)$  is the primitive of the nonlinearity  $f(s)$  and the parameters  $\varepsilon > 0$ ,  $0 < \mu < N$ . For  $\mu = 1$  and  $F(s) = \frac{1}{2}|s|^2$ , equation (SNS) was investigated by S.I. Pekar in [42] to study the quantum theory of a polaron at rest. In [28] P. Choquard suggested to use it as an approximation to the Hartree-Fock theory of one-component plasma. This equation was also proposed by R. Penrose in [36] as a model for self-gravitating particles and in that context it is known as the Schrödinger-Newton equation.

Notice that if  $u$  is a solution of the nonlocal equation (SNS) and  $x_0 \in \mathbb{R}^N$ , then the function  $v = u(x_0 + \varepsilon x)$  satisfies

$$-\Delta v + V(x_0 + \varepsilon x)v = \left[ \frac{1}{|x|^\mu} * F(v) \right] f(v) \quad \text{in } \mathbb{R}^N.$$

This suggests some convergence, as  $\varepsilon \rightarrow 0$ , of the family of solutions of (SNS) to a solution  $u_0$  of the limit problem

$$-\Delta v + V(x_0)v = \left[ \frac{1}{|x|^\mu} * F(v) \right] f(v) \quad \text{in } \mathbb{R}^N. \quad (1.1)$$

This is known as semi-classical limit for the nonlocal Choquard equation and we refer to [8, 9] for a survey on this topic. The study of semiclassical states for the Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N, \quad (1.2) \quad \boxed{\text{S.S}}$$

goes back to the pioneering work [24] by Floer and Weinstein. Since then, it has been studied extensively under various hypotheses on the potential and the nonlinearity, see for example [7, 16, 17, 24, 25, 26, 43, 44, 46, 48] and the references therein. In the study of semiclassical problems for local Schrödinger equations, the Lyapunov-Schmidt reduction method has been proved to be one of the most powerful tools. However, this technique relies greatly on the uniqueness and non-degeneracy of the ground states of the limit problem which is not completely settled for the ground states of the nonlocal Choquard equation

$$-\Delta u + u = \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \quad \text{in } \mathbb{R}^N. \quad (1.3) \quad \boxed{\text{CC}}$$

In [15, 33, 37], the authors investigated the qualitative properties of solutions and established the regularity, positivity, radial symmetry and decaying behavior at infinity. Moroz and Van Schaftingen in [38] established the existence of ground states with Berestycki-Lions type general nonlinearity. For  $N = 3$ ,  $\mu = 1$  and  $F(s) = \frac{1}{2}|s|^2$ , by proving the uniqueness and non-degeneracy of the ground states, Wei and Winter [47] constructed a family of solutions by a Lyapunov-Schmidt type reduction. In presence of non-constant electric and magnetic

potentials, Cingolani et.al. [14] showed the existence of family of solutions concentrating at regions localized by the minima of the potential. Moroz and Van Schaftingen [39] developed a nonlocal penalization technique to show that the equation  $(SNS)$  has a family of solutions concentrating at the local minimum of  $V$  provided  $V$  satisfies some additional assumptions at infinity. In [51], Yang and Ding considered the following equation

$$-\varepsilon^2 \Delta u + V(x)u = \left[ \frac{1}{|x|^\mu} * u^p \right] u^{p-1}, \quad \text{in } \mathbb{R}^3.$$

By using variational methods, for suitable parameters  $p, \mu$ , the authors obtained the existence of solutions. In [5], Alves and Yang proved the existence, multiplicity and concentration of solutions by penalization methods and Lusternik-Schnirelmann theory.

Let us recall the Hardy-Littlewood-Sobolev inequality, see [27], which will be frequently used throughout this paper:

**HLS** **Proposition 1.1** (Hardy-Littlewood-Sobolev inequality). *Let  $s, r > 1$  and  $0 < \mu < N$  with  $1/s + \mu/N + 1/r = 2$ ,  $f \in L^s(\mathbb{R}^N)$  and  $h \in L^r(\mathbb{R}^N)$ . There exists a sharp constant  $C(s, N, \mu, r)$ , independent of  $f, h$ , such that*

$$\int_{\mathbb{R}^N} \left[ \frac{1}{|x|^\mu} * f(x) \right] h(x) \leq C(s, N, \mu, r) |f|_s |h|_r.$$

Applying the Hardy-Littlewood-Sobolev inequality, we know

$$\int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u) \right] F(u)$$

is well defined if  $F(u) \in L^s(\mathbb{R}^N)$  for  $s > 1$  given by

$$\frac{2}{s} + \frac{\mu}{N} = 2.$$

This means that we must require

$$F(u) \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

Assume that  $F(u) = |u|^p$  and  $N \geq 3$ , to preserve the variational structure, the Sobolev Embedding Theorem implies that the exponent  $p$  must satisfy

$$\frac{2N-\mu}{N} \leq p \leq \frac{2N-\mu}{N-2}.$$

The confining exponents above play the role of critical exponents for the nonlocal Choquard equation in  $\mathbb{R}^N$ ,  $N \geq 3$ . To the authors' best knowledge, most of the works afore mentioned are set in  $\mathbb{R}^N$ ,  $N \geq 3$  with non-critical growth nonlinearities. Except for the case of the lower-critical exponent considered in [40], there is no results available on the existence and concentration of solutions for the nonlocal Choquard equation with upper-critical exponent  $\frac{2N-\mu}{N-2}$ .

The case  $N = 2$  is very special, as for bounded domains  $\Omega \subset \mathbb{R}^2$  the corresponding Sobolev embedding yields  $H_0^1(\Omega) \subset L^q(\Omega)$  for all  $q \geq 1$ , but  $H_0^1(\Omega) \not\subset L^\infty(\Omega)$ . For dimension  $N = 2$ ,

the Pohozaev-Trudinger-Moser inequality [34, 45] can be treated as a substitute of the Sobolev inequality as it establishes the following sharp maximal exponential integrability for functions in  $H_0^1(\Omega)$ :

$$\sup_{u \in H_0^1(\Omega) : \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\alpha u^2} \leq C|\Omega| \quad \text{if } \alpha \leq 4\pi,$$

for a positive constant which depends only on  $\alpha$ , where  $|\Omega|$  denotes Lebesgue measure of  $\Omega$ . As a consequence we say that a function  $f(s)$  has *critical exponential growth* if there exists  $\alpha_0 > 0$  such that

$$\lim_{|s| \rightarrow +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0, \quad \forall \alpha > \alpha_0, \quad \text{and} \quad \lim_{|s| \rightarrow +\infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty, \quad \forall \alpha < \alpha_0. \quad (1.4) \quad \boxed{\text{ecg}}$$

This definition of criticality was introduced by Adimurthi and Yadava [3], see also de Figueiredo, Miyagaki and Ruf [18]. The first version of the Pohozaev-Trudinger-Moser inequality in  $\mathbb{R}^2$  was established by Cao in [12], see also [2, 41, 13], and reads as follows

Trudinger-Moser

**Lemma 1.2.** *If  $\alpha > 0$  and  $u \in H^1(\mathbb{R}^2)$ , then*

$$\int_{\mathbb{R}^2} [e^{\alpha |u|^2} - 1] < \infty. \quad (1.5) \quad \boxed{\text{TM1}}$$

Moreover, if  $|\nabla u|_2^2 \leq 1$ ,  $|u|_2 \leq M < \infty$ , and  $\alpha < \alpha_0 = 4\pi$ , then there exists a constant  $C$ , which depends only on  $M$  and  $\alpha$ , such that

$$\int_{\mathbb{R}^2} [e^{\alpha |u|^2} - 1] \leq C(M, \alpha). \quad (1.6) \quad \boxed{\text{TM2}}$$

We refer the reader to [3, 30] for related problems and [13, 31, 52] for recent advances on this topic. Actually just a few papers deal with semiclassical states for local Schrödinger equations with critical exponential growth. In [19], do Ó and Souto proved the existence of solutions concentrating around local minima of  $V(x)$  which are not necessarily nondegenerate. For  $N$ -Laplacian equation in  $\mathbb{R}^N$ , Alves and Figueiredo [4] studied the multiplicity of semiclassical solutions with Rabinowitz type assumption on the potential. Recently, do Ó and Severo [20] and do Ó, Moameni and Severo [21] also studied a class of quasilinear Schrödinger equations in  $\mathbb{R}^2$  with critical exponential growth.

Hence it is quite natural to wonder if the existence and concentration results for local Schrödinger equations still hold for the nonlocal equation with critical growth in the sense of Pohozaev-Trudinger-Moser inequality. The purpose of this paper is two-fold: on the one hand we study the existence of nontrivial solution for the critical nonlocal equation with periodic potential, namely we consider the equation

$$-\Delta u + W(x)u = \left( \frac{1}{|x|^\mu} * F(u) \right) f(u), \quad \text{in } \mathbb{R}^2. \quad (1.7) \quad \boxed{\text{A1}}$$

Assume for the potential the following conditions:

(W<sub>1</sub>)  $W(x) \geq W_0 > 0$  in  $\mathbb{R}^2$  for some  $W_0 > 0$ ;

(W<sub>2</sub>)  $W(x)$  is a 1-periodic continuous function.

and for the nonlinearity  $f$  the following conditions:

- (f<sub>1</sub>) (i)  $f(s) = 0 \quad \forall s \leq 0, 0 \leq f(s) \leq Ce^{4\pi s^2}, \quad s \geq 0;$   
(ii)  $\exists s_0 > 0, M_0 > 0,$  and  $q \in (0, 1]$  such that  $0 < s^q F(s) \leq M_0 f(s), \forall |s| \geq s_0.$
- (f<sub>2</sub>) There exists  $p > \frac{2-\mu}{2}$  and  $C_p > 0$  such that  $f(s) \sim C_p s^p$ , as  $s \rightarrow 0.$
- (f<sub>3</sub>) There exists  $K > 1$  such that  $f(s)s > KF(s)$  for all  $s > 0$ , where  $F(t) = \int_0^t f(s)ds.$
- (f<sub>4</sub>)  $\lim_{s \rightarrow +\infty} \frac{sf(s)F(s)}{e^{8\pi s^2}} \geq \beta$ , with  $\beta > \inf_{\rho > 0} \frac{e^{\frac{4-\mu}{4}V_0\rho^2}}{16\pi^2\rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.$

Our first main result reads as follows,

Existence

**Theorem 1.3.** *Assume  $0 < \mu < 2$ , suppose that the potential  $V$  satisfies  $(W_1) - (W_2)$  and the nonlinearity  $f$  satisfies conditions  $(f_1) - (f_4)$ . Then equation (1.7) has a ground state solution in  $H^1(\mathbb{R}^2)$ .*

On the other hand, we establish the existence and concentration of semiclassical ground state solutions of the following equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-2} \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \quad \text{in } \mathbb{R}^2. \quad (1.8) \quad \boxed{\text{EC}}$$

Here we assume the following conditions on  $V$ :

- (V<sub>1</sub>)  $V(x) \geq V_0 > 0$  in  $\mathbb{R}^2$  for some  $V_0 > 0;$   
(V<sub>2</sub>)  $0 < \inf_{x \in \mathbb{R}^2} V(x) = V_0 < V_\infty = \liminf_{|x| \rightarrow \infty} V(x) < \infty.$

The condition (V<sub>2</sub>) was introduced by Rabinowitz in [46]. Hereafter, we will denote by

$$M = \{x \in \mathbb{R}^2 : V(x) = V_0\},$$

the minimum points set of  $V(x)$ .

We also assume that the nonlinearity satisfies the following

- (f<sub>5</sub>)  $s \rightarrow f(s)$  is strictly increasing on  $(0, +\infty).$

Then we prove our second main result.

T1

**Theorem 1.4.** *Suppose that the nonlinearity  $f(s)$  satisfies  $(f_1) - (f_5)$  and the potential function  $V(x)$  satisfies assumptions  $(V_1) - (V_2)$ . Then, for any  $\varepsilon > 0$  small, problem (1.8) has at least one positive ground state solution. Moreover, let  $u_\varepsilon$  denotes one of these positive solutions with  $\eta_\varepsilon \in \mathbb{R}^2$  its global maximum, then*

$$\lim_{\varepsilon \rightarrow 0} V(\eta_\varepsilon) = V_0.$$

Notation:

- $C, C_i$  denote positive constants.
- $B_R$  denotes the open ball centered at the origin with radius  $R > 0$ .
- $C_0^\infty(\mathbb{R}^2)$  denotes the space of the functions infinitely differentiable with compact support in  $\mathbb{R}^2$ .
- For a measurable function  $u$ , we denote by  $u^+$  and  $u^-$  its positive and negative parts respectively, given by

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

- $\|\cdot\|$  and  $|\cdot|_s$  denote the usual norms of the spaces  $H^1(\mathbb{R}^2)$  and  $L^s(\mathbb{R}^2)$  respectively.
- Let  $E$  be a real Hilbert space and  $I : E \rightarrow \mathbb{R}$  a functional of class  $\mathcal{C}^1$ . We say that  $\{u_n\} \subset E$  is a Palais-Smale (( $PS$ ) for short) sequence at  $c$  for  $I$  if  $\{u_n\}$  satisfies

$$I(u_n) \rightarrow c \quad \text{and} \quad I'(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Moreover,  $I$  satisfies the ( $PS$ ) condition at level  $c$ , if any ( $PS$ ) sequence  $\{u_n\}$  such that  $I(u_n) \rightarrow c$  possesses a convergent subsequence.

## 2 A critical nonlocal equation with periodic potential: proof of Theorem 1.3

In [6], Alves and Yang studied equation (1.7) under hypothesis ( $W_1$ ) and ( $W_2$ ) for the potential and the following conditions on the nonlinearity  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$ :

$$f(0) = 0, \quad \lim_{s \rightarrow 0} f'(s) = 0. \quad (f'_1)$$

It is of critical growth at infinity with  $\alpha_0 = 4\pi$ . Moreover, there exists  $C_0$  such that

$$|f'(s)| \leq C_0 e^{4\pi s^2}, \quad \forall s > 0. \quad (f'_2)$$

There exists  $\theta > 2$  such that

$$0 < \theta F(s) \leq 2f(s)s, \quad \forall s > 0, \quad (f'_3)$$

Furthermore, they suppose that there exists  $p > \frac{4-\mu}{2}$  such that

$$F(s) \geq C_p s^p, \quad \forall s > 0 \quad (f'_4)$$

where

$$C_p > \frac{\left[\frac{4\theta(p-1)}{(2-\mu)(\theta-2)}\right]^{\frac{p-1}{2}} S_p^p}{p^{\frac{p}{2}}}.$$

and

$$S_p = \inf_{u \in H^1(\mathbb{R}^2), u \neq 0} \frac{\left(\int_{\mathbb{R}^2} (|\nabla u|^2 + |W|_\infty |u|^2)\right)^{1/2}}{\left(\int_{\mathbb{R}^2} \left[\frac{1}{|x|^\mu} * |u|^p\right] |u|^p\right)^{\frac{1}{2p}}}.$$

Combining the above estimates with the Hardy-Littlewood-Sobolev inequality and some results due to P.L. Lions, the following existence result was obtained in [6].

AQ1

**Theorem 2.1.** *Suppose that conditions  $(f'_1) - (f'_4)$  hold. Then problem (1.7) has at least one ground state solution  $w$ .*

A key tool in [6] is assumption  $(f'_4)$  which enables one to obtain estimates of the Mountain-Pass level for the energy functional related to the nonlocal Choquard equation, for  $0 < \mu < 2$ ,

$$\begin{cases} -\Delta u + W(x)u = \left(\frac{1}{|x|^\mu} * F(u)\right)f(u), & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2) \\ u(x) > 0 \text{ for all } x \in \mathbb{R}^2. \end{cases} \quad (2.1) \quad \boxed{\text{A}}$$

Condition  $(f'_4)$  involves the explicit value of the best constant of the embedding  $H^1 \hookrightarrow L^p$ ,  $p \in (2, \infty)$ , which is so far unknown and still an open challenging problem. In terms of the nonlinear source, condition  $(f'_4)$  prescribe a global growth which can not be actually verified. This somehow affects possible further applications. The aim of this section is to overcome condition  $(f'_4)$  which we replace with the assumption  $(f_4)$ . For this purpose, we set

$$W_\rho := \sup_{|x| \leq \rho} W(x)$$

and

$$\mathcal{W} := \inf_{\rho > 0} \frac{e^{\frac{4-\mu}{4}W_\rho\rho^2}}{16\pi^2\rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.$$

Notice that if  $W(x)$  is continuous and  $(W_2)$  is satisfied, then  $W_\rho$  is a positive continuous function and  $\mathcal{W}$  can be attained by some  $\rho > 0$ . Moreover, it is worth to point out that assumption  $(f_1) - (ii)$  implies that for any  $\eta > 0$  there exists  $C_\eta > 0$  and  $s_\eta$  such that for all  $s \geq s_\eta$

$$\eta f(s) \geq F(s) \quad (2.2) \quad \boxed{\text{ARcond}}$$

and as  $s$  is large enough

$$F(s) \geq C_\eta e^{s^{\eta+1}}.$$

On the other hand,  $(f_1) - (ii)$  implies, for some  $\gamma > 0$ ,

$$F(s) \leq e^{\gamma s^2} - 1, \quad \text{for any } s > 0$$

which agrees with  $(f_2)$ . Notice also that assumptions  $(f_2)$  and  $(f_3)$  yield

$$K > \frac{4-\mu}{2} > 1.$$

Assumption  $(f_4)$  is inspired by [18, 52], but here we have the extra difficulty to handle integrals where both the two nonlinearities  $F(s)$  and  $sf(s)$  appear simultaneously. This situation forces us to assume condition  $(f_4)$  which is sharper than the following assumption of [18]

$$\lim_{s \rightarrow +\infty} \frac{F(s)}{e^{4\pi s^2}} \geq \gamma. \quad (2.3) \quad \boxed{\text{f}_4 \text{ bis}}$$

Actually, condition (2.3), combined with (2.2) implies

$$\lim_{s \rightarrow +\infty} \frac{sf(s)}{e^{4\pi s^2}} \geq \gamma \eta^{-1} \quad \text{for any } \eta > 0,$$

so that  $(f_4)$  is trivially satisfied for any choice of  $\gamma > 0$ . Finally, note that  $(f_4)$  together with (2.2) still imply

$$\lim_{s \rightarrow +\infty} \frac{sf(s)}{e^{4\pi s^2}} = +\infty,$$

but it may happen that

$$\lim_{s \rightarrow +\infty} \frac{F(s)}{e^{4\pi s^2}} = 0$$

in contrast with (2.3). This is the case, for instance, if

$$F(s) \sim \frac{e^{4\pi s^2}}{s} \text{ and } f(s) \sim 8\pi e^{4\pi s^2}, \quad s \rightarrow +\infty.$$

Since we are looking for positive solutions  $u \geq 0$ , from now on we assume  $f(s) = 0$  for  $s \leq 0$ . The energy functional associated with problem (2.1) is given by

$$\Phi_W(u) = \frac{1}{2} \|u\|_W^2 - \mathfrak{F}(u),$$

where

$$\mathfrak{F}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u) \right] F(u)$$

and

$$\|u\|_W := \left( \int_{\mathbb{R}^2} |\nabla u|^2 + W(x)|u|^2 \right)^{1/2}$$

Let  $E$  denote the space  $H^1(\mathbb{R}^2)$  equipped with the norm  $\|u\|_W$ , which is equivalent to the standard Sobolev norm.

As a consequence of Cao's inequality in Lemma 1.2,  $(f_2)$  and Hölder's inequality we have  $F(u) \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2)$  (note that  $(f_2)$  is weaker than  $(f'_1)$  of [6]), and the functional  $\Phi_W(u)$  is  $\mathcal{C}^1(E)$  thanks to a generalization of a Lions' result recently proved in [22]. Then the Mountain Pass geometry can be proved as in [6]. By the Ekeland Variational Principle [23], there exists a (PS) sequence  $(u_n) \subset E \subset H^1(\mathbb{R}^2)$  such that

$$\Phi'_W(u_n) \rightarrow 0, \quad \Phi_W(u_n) \rightarrow m_W,$$

where the Mountain Pass level  $m_W$  can be characterized by

$$0 < m_W := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_W(\gamma(t)) \tag{2.4} \quad \boxed{\mathfrak{m}}$$

with

$$\Gamma := \{ \gamma \in \mathcal{C}^1([0,1], E) : \gamma(0) = 0, \Phi_W(\gamma(1)) < 0 \}.$$

el-estimate

**Lemma 2.2.** *The mountain pass level  $m_W$  satisfies*

$$m_W < \frac{4-\mu}{8}.$$



*Proof.* It is enough to prove that there exists a function  $w \in E$ ,  $\|w\|_W = 1$ , such that

$$\max_{t \geq 0} \Phi_W(tw) < \frac{4 - \mu}{8}.$$

Let us introduce the following Moser type functions supported in  $B_\rho$  by

$$\bar{w}_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq \frac{\rho}{n}, \\ \frac{\log(\rho/|x|)}{\sqrt{\log n}}, & \frac{\rho}{n} \leq |x| \leq \rho, \\ 0, & |x| \geq \rho. \end{cases}$$

One has that

$$\begin{aligned} \|\bar{w}_n\|_W^2 &= \int_{B_\rho} |\nabla \bar{w}_n|^2 + \int_{B_\rho} W(x) |\bar{w}_n|^2 \\ &\leq \int_{\rho/n}^\rho \frac{dr}{r \log n} + W_\rho \int_0^{\rho/n} \log n \, r \, dr + W_\rho \int_{\rho/n}^\rho \frac{\log^2(\rho/r)}{\log n} \, r \, dr \\ &= 1 + \delta_n, \end{aligned}$$

where

$$\delta_n = W_\rho \rho^2 \left[ \frac{1}{4 \log n} - \frac{1}{4n^2 \log n} - \frac{1}{2n^2} \right] > 0. \quad (2.5)$$

And then, setting  $w_n = \bar{w}_n / \sqrt{1 + \delta_n}$ , we get  $\|w_n\|_W = 1$ .

We claim that there exists  $n$  such that

$$\max_{t \geq 0} \Phi_W(tw_n) < \frac{4 - \mu}{8}. \quad (2.6) \quad \boxed{\text{claim}}$$

Let us argue by contradiction and suppose this is not the case, so that for all  $n$  let  $t_n > 0$  be such that

$$\max_{t \geq 0} \Phi_W(tw_n) = \Phi_W(t_n w_n) \geq \frac{4 - \mu}{8}, \quad (2.7) \quad \boxed{\text{bycontr-assump}}$$

then  $t_n$  satisfies  $\frac{d}{dt} \Phi_W(tw_n)|_{t=t_n} = 0$  and

$$t_n^2 = \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(t_n w_n) \right] t_n w_n f(t_n w_n), \quad (2.8) \quad \boxed{t_n^2 =}$$

it follows from (2.7) that

$$t_n^2 \geq \frac{4 - \mu}{4}. \quad (2.9) \quad \boxed{\text{est-}t_n^2}$$

Let us estimate from below the quantity  $t_n^2$ . Taking advantage of equation (2.8), thanks to  $(f_4)$  we have for any  $\varepsilon > 0$ ,

$$sf(s)F(s) \geq (\beta - \varepsilon)e^{8\pi s^2} \quad \text{for all } s \geq s_\varepsilon \quad (2.10) \quad \boxed{\text{estimate-sfF}}$$

and thus

$$\begin{aligned}
t_n^2 &\geq \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} F(t_n w_n) dx \\
&= \int_{B_{\rho/n}} t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}} f\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}}\right) dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} F\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi}}\right) dx \\
&\geq (\beta - \varepsilon) e^{4t_n^2(1+\delta_n)^{-1} \log n} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} dx.
\end{aligned}$$

Notice that  $B_{\rho/n-|x|}(0) \subset B_{\rho/n}(x)$  since  $|x| \leq \rho/n$ , the last integral can be estimated as follows

$$\begin{aligned}
\int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{dx}{|x-y|^\mu} &= \int_{B_{\rho/n}} dx \int_{B_{\rho/n}(x)} \frac{dz}{|z|^\mu} \\
&\geq \int_{B_{\rho/n}} dx \int_{B_{\rho/n-|x|}} \frac{dz}{|z|^\mu} \\
&= \frac{2\pi}{2-\mu} \int_{B_{\rho/n}} \left(\frac{\rho}{n} - |x|\right)^{2-\mu} \\
&= \frac{4\pi^2}{2-\mu} \int_0^{\rho/n} \left(\frac{\rho}{n} - r\right)^{2-\mu} r dr \\
&= \frac{4\pi^2}{(2-\mu)(3-\mu)(4-\mu)} \left(\frac{\rho}{n}\right)^{4-\mu} \\
&= C_\mu \left(\frac{\rho}{n}\right)^{4-\mu},
\end{aligned} \tag{2.11}$$

where

$$C_\mu = \frac{4\pi^2}{(2-\mu)(3-\mu)(4-\mu)}.$$

Consequently, we obtain

$$\begin{aligned}
t_n^2 &\geq \frac{4\pi^2(\beta - \varepsilon)}{(2-\mu)(3-\mu)(4-\mu)} e^{4t_n^2(1+\delta_n)^{-1} \log n} \left(\frac{\rho}{n}\right)^{4-\mu} \\
&= \frac{4\pi^2(\beta - \varepsilon)\rho^{4-\mu}}{(2-\mu)(3-\mu)(4-\mu)} e^{\log n [4(1+\delta_n)^{-1} t_n^2 - (4-\mu)]}
\end{aligned}$$

which, recalling (2.9), means that  $t_n$  is bounded and yields

$$t_n^2 \longrightarrow \left(\frac{4-\mu}{4}\right)^+$$

as  $n$  goes to infinity. Moreover, as a byproduct we also have that for some  $C > 0$

$$\log n [4(1+\delta_n)^{-1} t_n^2 - (4-\mu)] \leq C,$$

that is

$$\frac{t_n^2}{1+\delta_n} = \frac{4-\mu}{4} + O\left(\frac{1}{\log n}\right). \tag{2.12} \quad \boxed{\text{est-}t_n^2\text{-bis}}$$

This estimate will be used to obtain a finer estimate than (2.9). Notice first that by  $(f_1)$  and  $(f_2)$  we have

$$F(s) \leq Cs^{\frac{4-\mu}{2}} + Mf(s) \leq Cs^{\frac{4-\mu}{2}} + C(e^{4\pi s^2} - 1). \quad (2.13) \quad \boxed{\text{estimate-F}}$$

Next define

$$A_n = \{y \in B_\rho : t_n w_n(y) > s_\varepsilon\} \quad \text{and} \quad B_n = B_\rho \setminus A_n,$$

where  $s_\varepsilon$  was introduced in (2.10). By (2.10) we know

$$\begin{aligned} t_n^2 &= \int_{\mathbb{R}^2} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) dy \\ &= \int_{B_\rho} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) dy \\ &= \int_{A_n} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) dy + \int_{B_n} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) dy. \end{aligned}$$

Combining the Hardy-Littlewood-Sobolev inequality with (2.13) one has

$$\begin{aligned} \int_{B_n} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) &\leq C \|F(t_n w_n)\|_{\frac{4}{4-\mu}} \|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}} \\ &\leq \left[ C \|t_n w_n\|_2 + C \left\{ \int_{\mathbb{R}^2} e^{4\pi \frac{4}{4-\mu} t_n^2 w_n^2} - 1 \right\}^{\frac{4-\mu}{4}} \right] \|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}}. \end{aligned} \quad (2.14) \quad \boxed{\text{finer-est1}}$$

By (2.12), since  $\|\nabla \bar{w}_n\|_2 = 1$  and  $\bar{w}_n^2 \leq 2\pi \log n$ , we obtain

$$\int_{\mathbb{R}^2} e^{4\pi \frac{4}{4-\mu} t_n^2 w_n^2} - 1 \leq \int_{B_\rho} e^{4\pi \frac{4}{4-\mu} t_n^2 w_n^2} \leq \int_{B_\rho} e^{4\pi(1+\frac{C}{\log n})\bar{w}_n^2} \leq \int_{B_\rho} C e^{4\pi \bar{w}_n^2} \leq C,$$

due to the Pohozaev-Trudinger-Moser inequality. Since  $t_n w_n \rightarrow 0$  a.e. and  $t_n w_n$  is bounded on  $B_n$ , applying the Lebesgue dominated convergence theorem, we obtain

$$\|\chi_{B_n} t_n w_n f(t_n w_n)\|_{\frac{4}{4-\mu}} \rightarrow 0,$$

as  $n \rightarrow \infty$ . Consequently,

$$t_n^2 = \int_{A_n} \left( \frac{1}{|x|^\mu} * F(t_n w_n) \right) t_n w_n f(t_n w_n) dy + o(1), \quad (2.15) \quad \boxed{\text{est-t}_n^2\text{-tris}}$$

where  $o(1)$  is actually positive.

Buying the same lines we can estimate the convolution term as follows

$$\begin{aligned} t_n^2 &\geq \int_{A_n} t_n w_n f(t_n w_n) dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx + \int_{A_n} t_n w_n f(t_n w_n) dy \int_{B_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\ &\geq \int_{A_n} t_n w_n f(t_n w_n) dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx + o(1). \end{aligned}$$

By the definition of  $w_n$ , we observe that

$$A_n = \{0 < |x| < \rho e^{-s_\varepsilon \sqrt{2\pi(1+\delta_n)\sqrt{\log n}}}\} \supset B_{\frac{\rho}{n}}^{\frac{\rho}{n}}, \quad (2.16) \quad \boxed{\text{calculA}_n}$$

then

$$\begin{aligned}
t_n^2 &\geq \int_{A_n} t_n w_n f(t_n w_n) dy \int_{A_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\
&\geq \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\
&\quad + \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\
&\quad + \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\
&\quad + \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} t_n w_n f(t_n w_n) dy \int_{\frac{\rho}{n} \leq |x| \cap x \in A_n} \frac{F(t_n w_n)}{|x-y|^\mu} dx \\
&:= I_1 + I_2 + I_3 + I_4 \\
&\geq I_1 \geq (\beta - \varepsilon) e^{8\pi t_n^2 w_n^2} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \frac{1}{|x-y|^\mu} dx
\end{aligned} \tag{2.17}$$

where we have used the fact that  $w_n$  is constant on the ball  $B_{\rho/n}$ . Thanks to (2.11) we have

$$\begin{aligned}
I_1 &\geq (\beta - \varepsilon) e^{4t_n^2(1+\delta_n)^{-1} \log n} \int_{|y| \leq \frac{\rho}{n}} dy \int_{|x| \leq \frac{\rho}{n}} \frac{1}{|x-y|^\mu} dx \\
&\geq (\beta - \varepsilon) C_\mu e^{4t_n^2(1+\delta_n)^{-1} \log n} \left(\frac{\rho}{n}\right)^{4-\mu}
\end{aligned} \tag{2.18}$$

and hence, recalling the definition of  $\delta_n$  in (2.5), we also have

$$\begin{aligned}
I_1 &\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{4t_n^2(1+\delta_n)^{-1} \log n - (4-\mu) \log n} \\
&\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{(4-\mu) \log n [(1+\delta_n)^{-1} - 1]} \\
&\geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-(4-\mu) \delta_n \log n} \\
&= (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-(4-\mu) W_\rho \rho^2 \left[ \frac{1}{4} - \frac{1}{4n^2} - \frac{\log n}{2n^2} \right]} \\
&\rightarrow (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-\frac{4-\mu}{4} W_\rho \rho^2},
\end{aligned}$$

as  $n \rightarrow +\infty$ . Combining the previous inequality with (2.17) and passing to the limit we get

$$\frac{4-\mu}{4} \geq (\beta - \varepsilon) C_\mu \rho^{4-\mu} e^{-\frac{4-\mu}{4} W_\rho \rho^2},$$

and since  $\varepsilon$  is arbitrary, in turn

$$\beta \leq \frac{4-\mu}{4C_\mu \rho^{4-\mu}} e^{\frac{4-\mu}{4} W_\rho \rho^2} = \frac{e^{\frac{4-\mu}{4} W_\rho \rho^2}}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)}.$$

However, by the definition of  $\mathcal{W}$ , since  $\beta > \mathcal{W}$  and  $(f_4)$ , there exists  $\rho > 0$  such that

$$\beta > \frac{e^{\frac{4-\mu}{4} W_\rho \rho^2}}{16\pi^2 \rho^{4-\mu}} \frac{(4-\mu)^2}{(2-\mu)(3-\mu)} \tag{2.19} \quad \boxed{\text{beta\_below}}$$

which is a contradiction and this concludes the proof.  $\square$

**Remark 2.3.** *It is worth to mention that actually estimate (2.17) can be improved, in the sense that the constant  $\mathcal{W}$  can be sharpened by exploiting  $I_2$ ,  $I_3$  and  $I_4$  and some additional technical growth assumptions on  $f(s)$ , which we omit here since we do not bring to effective advantages in this context.*

In the spirit of [52] we next prove that the limit of a Palais-Smale sequence for  $\Phi_V$  yields a weak solution to (2.1).

**lem-PS**

**Lemma 2.4.** *Assume  $(W_1) - (W_2)$ ,  $(f_1) - (f_4)$  and let  $\{u_n\} \subset E$  be a Palais-Smale sequence for  $\Phi_W$ , i.e.*

$$\Phi_W(u_n) \rightarrow c \quad \text{and} \quad \Phi'_W(u_n) \rightarrow 0 \quad \text{in } E^*, \quad \text{as } n \rightarrow +\infty.$$

*Then there exists  $u \in E$  such that, up to subsequence,  $u_n \rightharpoonup u$  weakly in  $E$ ,*

$$\left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n) \rightarrow \left[ \frac{1}{|x|^\mu} * F(u) \right] F(u), \quad \text{in } L^1_{loc}(\mathbb{R}^2) \quad (2.20) \quad \text{convFF}$$

*and  $u$  is a weak solution of (2.1).*

*Proof.* By hypothesis we have

$$\frac{1}{2} \|u_n\|_W^2 - \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n) \rightarrow c \quad (2.21) \quad \text{convPhi}$$

as well as

$$\left| \int_{\mathbb{R}^2} \nabla u_n \nabla v + W u_n v - \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) v \right| \leq \tau_n \|v\|_W$$

for all  $v \in E$ , where  $\tau_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Taking  $v = u_n$  in (2.22) we obtain

$$\left| \|u_n\|_W^2 - \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] u_n f(u_n) \right| \leq \tau_n \|u_n\|_W. \quad (2.22) \quad \text{convPhi''}$$

By  $(f_1)$  that for any  $s > 0$  one has  $sf(s) \geq KF(s)$ . Then,

$$\int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] u_n f(u_n) \geq K \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n)$$

and so

$$\frac{1}{2} \left( 1 - \frac{1}{K} \right) \|u_n\|_W^2 \leq \Phi_W(u_n) - \frac{1}{2K} \langle \Phi'_W(u_n), u_n \rangle \leq \frac{c}{2} + \frac{\tau_n}{2K} \|u_n\|_W$$

which implies that  $\|u_n\|_W$  is bounded. As a consequence we have from (2.21) and (2.22) that

$$\int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n) \leq C, \quad \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] u_n f(u_n) \leq C \quad (2.23) \quad \text{bound}$$

with  $C$  independent of  $n$ . Moreover,  $u_n \rightharpoonup u$ ,  $u_n \rightarrow u$  in  $L^q_{loc}(\mathbb{R}^2)$  for any  $1 \leq q < \infty$  and  $u_n \rightarrow u$  a.e. in  $\mathbb{R}^2$ .

Next let us prove (2.20), that is,

$$\left| \int_{\Omega} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] F(u_n) dx - \int_{\Omega} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) dx \right| \rightarrow 0, \quad \forall \Omega \subset \subset \mathbb{R}^2$$

This can be done as in [18, Lemma 2.1]. Indeed, since  $u \in H^1(\mathbb{R}^2)$ , then  $\left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) \in L^1(\mathbb{R}^2)$ , so that

$$\lim_{M \rightarrow \infty} \int_{\{u \geq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) dx = 0.$$

Let  $C$  be the constant in (2.23) and  $M_0$  the constant in  $(f_1)$ : for any  $\delta > 0$  we can choose  $M > \max\{(CM_0/\delta)^{q+1}, s_0\}$  such that

$$0 \leq \int_{\{u \geq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) dx < \delta.$$

From (2.23) and  $(f_1)(ii)$  we also have

$$0 \leq \int_{\{u_n \geq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] F(u_n) dx \leq \frac{M_0}{M^{q+1}} \int_{\{u_n \geq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] u_n f(u_n) dx < \delta,$$

then we obtain

$$\begin{aligned} & \left| \int_{\Omega} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] F(u_n) dx - \int_{\Omega} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) dx \right| \leq \\ & 2\delta + \left| \int_{\Omega \cap \{u_n \leq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] F(u_n) dx - \int_{\Omega \cap \{u \leq M\}} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) dx \right|. \end{aligned}$$

It remains then to prove that

$$\int_{|u_n| \leq M} \left[ \frac{1}{|x|^{\mu}} * F(u_n) \right] F(u_n) \chi_{\Omega} dx \rightarrow \int_{|u| \leq M} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] F(u) \chi_{\Omega} dx \quad (2.24) \quad \boxed{\text{equivts}}$$

as  $n \rightarrow +\infty$ , for any fixed  $M > \max\{(CM_0/\delta)^{q+1}, s_0\}$ . Let us observe that as  $K \rightarrow +\infty$

$$\int_{|u| \leq M} \int_{|u| \leq K} \left[ \frac{F(u(y))}{|x-y|^{\mu}} \right] dy F(u(x)) \chi_{\Omega}(x) dx \rightarrow \int_{|u| \leq M} \left[ \frac{1}{|x|^{\mu}} * F(u) \right] dy F(u) \chi_{\Omega} dx.$$

Let  $C$  be the constant in (2.23) and choose  $K \geq \max\{(CM_0/\delta)^{q+1}, s_0\}$  such that

$$\int_{|u| \leq M} \int_{|u| \geq K} \left[ \frac{F(u(y))}{|x-y|^{\mu}} \right] dy F(u(x)) dx \leq \delta.$$

By  $(f_1)(ii)$  one has

$$\begin{aligned}
& \int_{|u_n| \leq M} \int_{|u_n| \geq K} \left[ \frac{F(u_n(y))}{|x-y|^\mu} \right] F(u_n(x)) \chi_\Omega(x) dx \\
& \leq \frac{1}{K^{q+1}} \int_{|u_n| \leq M} \int_{|u_n| \geq K} \left[ \frac{u_n^{q+1} F(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_\Omega dx \\
& \leq \frac{M_0}{K^{q+1}} \int_{|u_n| \leq M} \int_{|u_n| \geq K} \left[ \frac{u_n f(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_\Omega dx \\
& \leq \frac{M_0}{K^{q+1}} \int_{|u_n| \leq M} \int_{|u_n| \geq K} \left[ \frac{u_n f(u_n)}{|x-y|^\mu} \right] dy F(u_n) dx \\
& = \frac{M_0}{K^{q+1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[ \frac{F(u_n)}{|x-y|^\mu} \right] dy u_n f(u_n) dx \\
& \leq \delta,
\end{aligned}$$

then we can see that

$$\left| \int_{|u| \leq M} \int_{|u| \geq K} \left[ \frac{F(u)}{|x-y|^\mu} \right] dy F(u) \chi_\Omega - \int_{|u_n| \leq M} \int_{|u_n| \geq K} \left[ \frac{F(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_\Omega \right| \leq 2\delta.$$

In order to prove (2.24) it remains to verify that as  $n \rightarrow +\infty$  there holds

$$\left| \int_{|u| \leq M} \int_{|u| \leq K} \left[ \frac{F(u)}{|x-y|^\mu} \right] dy F(u) \chi_\Omega - \int_{|u_n| \leq M} \int_{|u_n| \leq K} \left[ \frac{F(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_\Omega \right| \rightarrow 0$$

for any fixed  $K, M > 0$ . This is a consequence of the Lebesgue's dominated convergence theorem: indeed,

$$\int_{|u_n| \leq K} \left[ \frac{F(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_{\{\Omega \cap |u_n| \leq M\}} \rightarrow \int_{|u| \leq K} \left[ \frac{F(u)}{|x-y|^\mu} \right] dy F(u) \chi_{\{\Omega \cap |u| \leq M\}} \quad \text{a.e.}$$

and by  $(f_2)$  we know there exists a constant  $C_{M,K}$  depends of  $M, K$  such that

$$\begin{aligned}
& \int_{|u_n| \leq K} \left[ \frac{F(u_n)}{|x-y|^\mu} \right] dy F(u_n) \chi_{\{\Omega \cap |u_n| \leq M\}} \\
& \leq C_{M,K} \int_{|u_n| \leq K} \left[ \frac{u_n^{p+1}}{|x-y|^\mu} \right] dy u_n^{p+1} \chi_{\{\Omega \cap |u_n| \leq M\}} \\
& \leq C_{M,K} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * u_n^{p+1} \right] u_n^{p+1} \chi_\Omega \rightarrow C_{M,K} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * u^{p+1} \right] u^{p+1} \chi_\Omega
\end{aligned}$$

as  $n \rightarrow \infty$ , applying the Hardy-Sobolev-Littlewood inequality, since  $u_n \rightarrow u$  in  $L_{loc}^s$  for all  $s \geq 1$ . Hence the proof of (2.20) is now complete.

Let us now prove that the weak limit  $u$  yields actually a weak solution to (2.1), namely that

$$\int_{\mathbb{R}^2} \nabla u \nabla \varphi + W(x) u \varphi - \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \varphi = 0 \quad (2.25) \quad \boxed{\text{weaklim}}$$

for all  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ . Since  $\{u_n\}$  is a  $(PS)_{m_V}$  sequence, for all  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ , we know that

$$\int_{\mathbb{R}^2} \nabla u_n \nabla \varphi + W(x) u_n \varphi - \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) \varphi \rightarrow 0,$$

as  $n \rightarrow \infty$ . Since  $u_n \rightharpoonup u$  in  $E$ , we just need to prove that, as  $n \rightarrow \infty$

$$\int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) \varphi \rightarrow \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \varphi \quad (2.26) \quad \boxed{\text{weak*conv}}$$

for all  $\varphi \in C_c^\infty(\mathbb{R}^2)$ .

Let  $\Omega$  be any compact subset of  $\mathbb{R}^2$ , we claim that there exists  $C(\Omega)$  such that

$$\int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] \frac{f(u_n)}{1+u_n} dx \leq C(\Omega). \quad (2.27) \quad \boxed{\text{boundbis}}$$

In fact, let

$$v_n = \frac{\varphi}{1+u_n},$$

where  $\varphi$  is a smooth function compactly supported in  $\Omega' \supset \Omega$ ,  $\Omega'$  compact, such that  $0 \leq \varphi \leq 1$  and  $\varphi \equiv 1$  in  $\Omega$ . Direct computation shows that

$$\begin{aligned} \|v_n\|_W^2 &= \int_{\mathbb{R}^2} |\nabla v_n|^2 + W(x) v_n^2 \\ &= \int_{\mathbb{R}^2} \left| \frac{\nabla \varphi}{1+u_n} - \varphi \frac{\nabla u_n}{(1+u_n)^2} \right|^2 + W \frac{\varphi^2}{(1+u_n)^2} \\ &\leq \int_{\mathbb{R}^2} \frac{|\nabla \varphi|^2}{(1+u_n)^2} + 2 \frac{\nabla \varphi \nabla u_n}{1+u_n} + \varphi^2 \frac{|\nabla u_n|^2}{(1+u_n)^4} + W \varphi^2 \\ &\leq 2 \|\varphi\|_W^2 + 2 \|u_n\|_W^2, \end{aligned}$$

which means that  $v_n \in E$ . Choose  $v_n$  as test function in (2.22), then

$$\begin{aligned} \int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] \frac{f(u_n)}{1+u_n} dx &\leq \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) \frac{\varphi}{1+u_n} \\ &\leq \int_{\mathbb{R}^2} |\nabla u_n|^2 \frac{\varphi}{(1+u_n)^2} + \frac{\nabla u_n \nabla \varphi}{1+u_n} + W u_n \frac{\varphi}{1+u_n} + \tau_n \|v_n\|_W \\ &\leq \int_{\mathbb{R}^2} |\nabla u_n|^2 \frac{\varphi}{(1+u_n)^2} + \frac{\nabla u_n \nabla \varphi}{1+u_n} + W u_n \frac{\varphi}{1+u_n} + 2\tau_n \|u_n\|_W + 2\tau_n \|\varphi\|_W \\ &\leq \|\nabla u_n\|_2^2 + C_\varphi \|\nabla u_n\|_2 + \int_{\Omega'} W u_n + 2\tau_n \|u_n\|_W + 2\tau_n \|\varphi\|_W. \end{aligned}$$

Since  $W(x)$  is bounded,  $u_n$  is bounded in  $H^1$  and  $u_n \rightarrow u$  in  $L^1(\Omega')$  we easily deduce (2.27).

Now define

$$\xi_n := \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n),$$

we can observe that

$$\begin{aligned} &\int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) dx \\ &\leq 2 \int_{\{u_n < 1\} \cap \Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] \frac{f(u_n)}{1+u_n} dx + \int_{\{u_n > 1\} \cap \Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] u_n f(u_n) dx \\ &\leq 2 \int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] \frac{f(u_n)}{1+u_n} dx + \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] u_n f(u_n) dx. \end{aligned}$$



Combining (2.27) and (2.23), it is easy to see that  $\xi_n$  is uniformly bounded in  $L^1(\Omega)$  with

$$\int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) dx \leq 2C(\Omega) + C.$$

Finally, consider the sequence of measures  $\mu_n$  with density  $\xi_n = \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n)$ , that is

$$\mu_n(E) := \int_E \xi_n dx = \int_E \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) dx \quad \text{for any measurable } E \subset \Omega.$$

Since  $\|\xi_n\|_1 \leq C(\Omega)$  and  $\Omega$  is bounded, the measures  $\mu_n$  have uniformly bounded total variation. Then, by weak\*-compactness, up to a subsequence,  $\mu_n \rightharpoonup^* \mu$  for some measure  $\mu$ ,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \xi_n \varphi dx = \lim_{n \rightarrow \infty} \int_{\Omega} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) \varphi dx = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Now recall that  $u_n$  is a (PS) sequence, so that in particular (2.22) holds and hence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \nabla u_n \nabla \varphi + W(x) u_n \varphi = \int_{\Omega} \varphi d\mu, \quad \forall \varphi \in C_c^\infty(\Omega),$$

which implies that  $\mu$  is absolutely continuous with respect to the Lebesgue measure. Then, by the Radon-Nicodym theorem, there exists a function  $\xi \in L^1(\Omega)$  such that

$$\int_{\Omega} \varphi d\mu = \int_{\Omega} \varphi \xi dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Since this holds for any compact set  $\Omega \subset \mathbb{R}^2$ , we have that there exists a function  $\xi \in L_{loc}^1(\mathbb{R}^2)$  such that

$$\int_{\mathbb{R}^2} \varphi d\mu = \lim_n \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n) \varphi dx = \int_{\mathbb{R}^2} \varphi \xi dx, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^2),$$

where  $\xi = \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u)$  and the proof is complete. □

**Proof of Theorem 1.3.** As proved in [6, Lemma 2.1], the functional  $\Phi_W$  satisfies the Mountain Pass geometry, then there exists a  $(PS)_{m_W}$  sequence  $\{u_n\}$ . By Lemma 2.4, up to a subsequence,  $\{u_n\}$  weakly converges to a weak solution  $u$  of (2.1): it remains only to prove that  $u$  is non-trivial. Let us suppose by contradiction that  $u \equiv 0$ . Since  $\{u_n\}$  is bounded, we have either  $\{u_n\}$  is vanishing, that is, for any  $r > 0$

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_r(y)} |u_n|^2 = 0$$

or it is non-vanishing, i.e. there exist  $r, \delta > 0$  and a sequence  $\{y_n\} \subset \mathbb{Z}^2$  such that

$$\lim_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^2 \geq \delta.$$

If  $\{u_n\}$  is vanishing, by Lions' concentration-compactness result we have

$$u_n \rightarrow 0 \quad \text{in } L^s(\mathbb{R}^2) \quad \forall s > 2, \tag{2.28} \quad \boxed{\text{Lions}}$$

as  $n \rightarrow \infty$ . In this case we claim that

$$\left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n) \rightarrow 0 \quad \text{in } L^1(\mathbb{R}^2), \quad (2.29) \quad \boxed{\text{conv0}}$$

as  $n \rightarrow \infty$ . In fact, we need only to repeat the proof of (2.20) in Lemma 2.4 without restricting necessarily to compact sets. Apply the Hardy-Sobolev-Littlewood inequality we notice that

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * u_n^{p+1} \right] u_n^{p+1} \right| \leq C |u_n|_{\frac{4}{4-\mu}(p+1)}^{2(p+1)} \rightarrow 0$$

as  $n \rightarrow \infty$ , since  $\frac{4}{4-\mu}(p+1) > 2$  and (2.28) holds. Since  $\{u_n\}$  is a  $(PS)_{m_W}$  sequence with  $m_W < \frac{4-\mu}{8}$ , it follows that

$$\lim_{n \rightarrow +\infty} \|u_n\|_W^2 = 2m_W < \frac{4-\mu}{4}$$

Then there exist a sufficiently small  $\delta > 0$  and  $K > 0$  such that

$$\|u_n\|_W^2 \leq \frac{4-\mu}{4}(1-\delta), \quad \forall n > K. \quad (2.30) \quad \boxed{\text{conv-norm}}$$

Using again the Hardy-Sobolev-Littlewood inequality we have

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n)u_n \right| \leq C |F(u_n)|_{\frac{4}{4-\mu}} |f(u_n)u_n|_{\frac{4}{4-\mu}}.$$

Combining  $(f_1)$  with  $(f_2)$ , for any  $\varepsilon > 0$ ,  $p > 1$  and  $\beta > 1$ , there exists  $C(\varepsilon, p, \beta) > 0$  such that

$$|f(s)| \leq \varepsilon |s|^{\frac{2-\mu}{2}} + C(\varepsilon, p, \beta) |s|^{p-1} [e^{\beta 4\pi s^2} - 1] \quad \forall s \in \mathbb{R}.$$

Then,

$$|f(u_n)u_n|_{\frac{4}{4-\mu}} \leq \varepsilon |u_n|_2^{\frac{4-\mu}{2}} + C(\varepsilon, p, \beta) |u_n|_{\frac{4pt'}{4-\mu}}^{\frac{4-\mu}{4t'}} \left( \int_{\mathbb{R}^2} [e^{(\frac{4\beta t}{4-\mu} \|u_n\|_W^2 4\pi \frac{u_n^2}{\|u_n\|_W^2})} - 1] \right)^{\frac{4-\mu}{4t}}$$

where  $t, t' > 1$  satisfying  $\frac{1}{t} + \frac{1}{t'} = 1$ . In order to conclude by means of [41] by do Ó and Adachi-Tanaka inequality [2] it is enough to choose  $\beta, t > 1$  close to 1 such that  $\frac{4\beta t}{4-\mu} \|u_n\|_W^2 < 1$ , namely

$$1 < \beta t < \frac{1}{1-\delta},$$

we deduce that

$$\left( \int_{\mathbb{R}^2} [e^{(\frac{4\beta t}{4-\mu} \|u_n\|_W^2 4\pi \frac{u_n^2}{\|u_n\|_W^2})} - 1] \right)^{\frac{4-\mu}{4t}} \leq \left( \int_{\mathbb{R}^2} [e^{(\frac{4\beta mt}{4-\mu} 4\pi \frac{u_n^2}{\|u_n\|_W^2})} - 1] \right)^{\frac{4-\mu}{4t}} \leq C_1 \quad \forall n > K,$$

for some  $C_1 > 0$ . Then,

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n)u_n \right| \leq \varepsilon^2 |u_n|_2^{4-\mu} + C_2 |u_n|_{\frac{4pt'}{4-\mu}}^{\frac{4-\mu}{2t'}}.$$

Since  $t > 1$  is close to 1, we have that  $\frac{4pt'}{4-\mu} > 2$ . By (2.28), we have

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n)u_n \right| \rightarrow 0$$

as  $n \rightarrow \infty$ . Recalling that  $\{u_n\}$  is a  $(PS)_{m_W}$  sequence,  $u_n \rightarrow 0$  in  $E$ , and so  $\Phi_W(u_n) \rightarrow 0$  which implies  $m_W = 0$ , which is a contradiction. Therefore the vanishing case dose not hold.

Let us now consider the non vanishing case and define  $v_n := u_n(\cdot - y_n)$ , then

$$\int_{B_r(0)} |v_n|^2 \geq \delta \quad (2.31) \quad \boxed{\text{nonvan}}$$

By the periodicity assumption,  $\Phi_W$  and  $\Phi'_W$  are both invariant by  $\mathbb{Z}^2$  translations, so that  $\{v_n\}$  is again a  $(PS)_{m_W}$  sequence. Then  $v_n \rightharpoonup v$  in  $E$ , with  $v \neq 0$  by using (2.31), since  $v_n \rightarrow v$  in  $L^2_{loc}(\mathbb{R}^2)$ . Thereby,  $v$  is a nontrivial critical point of  $\Phi_W$  and  $\Phi_W(v) = m_W$ , which completes the proof of the theorem.

### 3 Semiclassical states for the nonlocal Schrödinger equation

Performing the scaling  $u(x) = v(\epsilon x)$  one easily sees that problem (1.8) is equivalent to

$$-\Delta u + V(\epsilon x)u = \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u). \quad (SNS^*)$$

For  $\epsilon > 0$ , we define the following Hilbert space

$$E_\epsilon = \left\{ u \in E : \int_{\mathbb{R}^2} V(\epsilon x) |u|^2 < \infty \right\}$$

endowed with the norm

$$\|u\|_\epsilon := \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(\epsilon x) |u|^2) \right)^{1/2}.$$

The energy functional associated to equation  $(SNS^*)$  is given by

$$I_\epsilon(u) = \frac{1}{2} \|u\|_\epsilon^2 - \mathfrak{F}(u)$$

and

$$\langle I'_\epsilon(u), \varphi \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + V(\epsilon x) u \varphi) - \mathfrak{F}'(u)[\varphi], \quad \forall u, \varphi \in E.$$

Let  $\mathcal{N}_\epsilon$  be the Nehari manifold associated to  $I_\epsilon$ , that is,

$$\mathcal{N}_\epsilon = \left\{ u \in E_\epsilon : u \neq 0, \langle I'_\epsilon(u), u \rangle = 0 \right\}.$$

The following Lemma states that the Nehari manifold  $\mathcal{N}_\epsilon$  is bounded away from 0.

LN **Lemma 3.1.** *Suppose that conditions  $(f_1) - (f_3)$  hold. Then there exists  $\alpha > 0$ , independent of  $\epsilon$ , such that*

$$\|u\|_\epsilon \geq \alpha, \quad \forall u \in \mathcal{N}_\epsilon. \quad (3.1) \quad \boxed{\text{alpha2}}$$

*Proof.* For any  $\delta > 0$ ,  $p > 1$  and  $\beta > 1$ , there exists  $C_\delta > 0$  such that

$$F(s) < \frac{1}{K} f(s)s \leq \delta s^{\frac{4-\mu}{2}} + C(\delta, p, \beta) s^p [e^{\beta 4\pi s^2} - 1], \forall s \in \mathbb{R},$$

it follows

$$|F(u)|_{\frac{4}{4-\mu}} \leq C |f(u)u|_{\frac{4}{4-\mu}} \leq \delta C |u|_2^{\frac{4-\mu}{2}} + C(\delta, p, \beta) |u^p [e^{\beta 4\pi u^2} - 1]|_{\frac{4}{4-\mu}}. \quad (3.2) \quad \boxed{\text{mp1}}$$

Since the imbedding  $E_\varepsilon \hookrightarrow L^p(\mathbb{R}^2)$  is continuous for any  $p \in (2, +\infty)$ , we know there exists a constant  $C_1$  such that

$$\begin{aligned} \int_{\mathbb{R}^2} |u|^{\frac{4p}{4-\mu}} [e^{\beta 4\pi u^2} - 1]^{\frac{4}{4-\mu}} &\leq \left( \int_{\mathbb{R}^2} |u|^{\frac{8p}{4-\mu}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} [e^{\beta 4\pi u^2} - 1]^{\frac{4}{4-\mu}} \right)^{\frac{1}{2}} \\ &\leq C_1 \|u\|_\varepsilon^{\frac{4p}{4-\mu}} \left( \int_{\mathbb{R}^2} [e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1] \right)^{\frac{1}{2}}. \end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2} [e^{(\frac{4\beta}{4-\mu} 4\pi u^2)} - 1] = \int_{\mathbb{R}^2} [e^{(\frac{4\beta}{4-\mu} \|u\|_\varepsilon^2 4\pi \frac{u^2}{\|u\|_\varepsilon^2})} - 1],$$

then, fixing  $\xi \in (0, 1)$  and making  $\frac{4\beta}{4-\mu} \|u\|_\varepsilon^2 = \xi < 1$ , Lemma 1.2 implies that there exists a constant  $C_2$  such that

$$\int_{\mathbb{R}^2} [e^{(\xi 4\pi \frac{u^2}{\|u\|_\varepsilon^2})} - 1] \leq C_2.$$

thus, by (3.2), we know there exists  $C_3$  such that

$$|F(u)|_{\frac{4}{4-\mu}} \leq \delta \|u\|_\varepsilon^{\frac{4-\mu}{2}} + C_3 \|u\|_\varepsilon^p.$$

By the Hardy-Littlewood-Sobolev inequality, if  $\|u\|_\varepsilon^2 = \frac{\xi(4-\mu)}{4\beta}$ , there holds

$$\mathfrak{F}'(u)[u] \leq \delta^2 C_4 \|u\|_\varepsilon^{4-\mu} + C_4 \|u\|_\varepsilon^{2p}.$$

Since  $u \in \mathcal{N}_\varepsilon$ , there holds

$$\|u\|_\varepsilon^2 = \mathfrak{F}'(u)[u],$$

and so

$$\|u\|_\varepsilon^2 \leq \delta^2 C_5 \|u\|_\varepsilon^{4-\mu} + C_5 \|u\|_\varepsilon^{2p},$$

then the conclusion follows immediately.  $\square$

Next we show that the functional  $I_\varepsilon$  satisfies the Mountain Pass geometry.

**mountain:1** **Lemma 3.2.** *Suppose that conditions  $(f_1) - (f_3)$  hold, then*

- (i) *There exist  $\rho, \delta_0 > 0$  such that  $I_\varepsilon|_S \geq \delta_0 > 0$  for all  $u \in S = \{u \in E_\varepsilon : \|u\|_\varepsilon = \rho\}$ ;*
- (ii) *There is  $e$  with  $\|e\|_\varepsilon > \rho$  such that  $I_\varepsilon(e) < 0$ .*

*Proof.* The proof of (i) easily follows the line of Lemma 3.1, and so we only prove (ii). Fixed  $u_0 \in E_\varepsilon$  with  $u_0^+(x) = \max\{u_0(x), 0\}$ , we set

$$w(t) = \mathfrak{F}\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right) > 0, \quad \text{for } t > 0.$$

By the Ambrosetti-Rabinowitz condition  $(f_3)$  we know

$$\frac{w'(t)}{w(t)} \geq \frac{2K}{t} \quad \text{for } t > 0.$$

Integrate this over  $[1, s\|u_0\|_\varepsilon]$  with  $s > \frac{1}{\|u_0\|_\varepsilon}$  to get

$$\mathfrak{F}(su_0) \geq \mathfrak{F}\left(\frac{u_0}{\|u_0\|_\varepsilon}\right) \|u_0\|_\varepsilon^{2K} s^{2K}.$$

Therefore

$$I_\varepsilon(su_0) \leq C_1 s^2 - C_2 s^{2K} \quad \text{for } s > \frac{1}{\|u_0\|_\varepsilon}.$$

Since  $K > 1$ , (ii) follows taking  $e = su_0$  and  $s$  large enough.  $\square$

By the Ekeland Variational Principle [23] we know there is a  $(PS)_{c_\varepsilon}$  sequence  $(u_n) \subset E$ , i.e.

$$I'_\varepsilon(u_n) \rightarrow 0, \quad I_\varepsilon(u_n) \rightarrow c_\varepsilon,$$

where  $c_\varepsilon$  is defined by

$$0 < c_\varepsilon := \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tu) \quad (3.3) \quad \boxed{\text{m1}}$$

and moreover there is a constant  $c > 0$  independent of  $\varepsilon$  such that  $c_\varepsilon > c > 0$ . Using assumption  $(f_5)$ , for each  $u \in E_\varepsilon \setminus \{0\}$ , there is an unique  $t = t(u)$  such that

$$I_\varepsilon(t(u)u) = \max_{s \geq 0} I_\varepsilon(su) \quad \text{and } t(u)u \in \mathcal{N}_\varepsilon.$$

Then it is standard to see (see [50]) that the minimax value  $c_\varepsilon$  can be characterized by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u). \quad (3.4) \quad \boxed{\text{m2}}$$

EML **Lemma 3.3.** *Suppose that assumptions  $(f_1) - (f_5)$ ,  $(V_1)$  and  $(V_2)$  hold. Let  $c_\varepsilon$  be the minimax value defined in (3.3), then there holds*

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon = m_{V_0},$$

where  $m_{V_0}$  is the minimax value defined in (2.4) with  $W(x) \equiv V_0$ . Hence, by Lemma 2.2, there is  $\varepsilon_0 > 0$  such that

$$c_\varepsilon < \frac{4 - \mu}{8}, \quad \forall \varepsilon \in [0, \varepsilon_0).$$

Moreover, since  $m_{V_0} < m_{V_\infty}$ , we also have

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon \leq m_{V_\infty}.$$

*Proof.* Let  $w \in E$  be the ground state solution obtained in Theorem 1.3, then there holds

$$\int_{\mathbb{R}^2} (|\nabla w|^2 + V_0|w|^2) = \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(w) \right] f(w)w.$$

In what follows, given  $\delta > 0$ , we fix  $w_\delta \in C_0^\infty(\mathbb{R}^2)$  verifying

$$w_\delta \in \mathcal{N}_{V_0}, \quad w_\delta \rightarrow w \text{ in } E \text{ and } \Phi_{V_0}(w_\delta) < m_{V_0} + \delta. \quad (3.5) \quad \boxed{\text{ESc1}}$$

Now, choose  $\eta \in C_0^\infty(\mathbb{R}^2, [0, 1])$  be such that  $\eta = 1$  on  $B_1(0)$  and  $\eta = 0$  on  $\mathbb{R}^2 \setminus B_2(0)$ , let us define  $v_n(x) = \eta(\varepsilon_n x)w_\delta(x)$ , where  $\varepsilon_n \rightarrow 0$ . Clearly

$$v_n \rightarrow w_\delta \text{ in } E, \text{ as } n \rightarrow +\infty.$$

From the definition of  $\mathcal{N}_\varepsilon$ , we know that there exists unique  $t_n$  such that  $t_n v_n \in \mathcal{N}_{\varepsilon_n}$ . Consequently,

$$c_{\varepsilon_n} \leq I_{\varepsilon_n}(t_n v_n) = \frac{t_n^2}{2} \int_{\mathbb{R}^2} (|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2) - \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(t_n v_n) \right] F(t_n v_n).$$

Observe that

$$\langle I'_{\varepsilon_n}(t_n v_n), t_n v_n \rangle = 0,$$

or equivalently,

$$\begin{aligned} t_n^2 \int_{\mathbb{R}^2} (|\nabla v_n|^2 + V(\varepsilon_n x)|v_n|^2) &= \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(t_n v_n) \right] f(t_n v_n) t_n v_n \\ &\geq C t_n^{2K} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * |v_n|^K \right] |v_n|^K \end{aligned} \quad (3.6) \quad \boxed{\text{ESc}}$$

which means  $\{t_n\}$  is bounded and thus, up to subsequence, we may assume that  $t_n \rightarrow t_0 \geq 0$ . Notice that there is a constant  $c > 0$  independent of  $\varepsilon$  such that  $c_{\varepsilon_n} > c > 0$ . Then, this information implies that  $t_0 > 0$ . Take a limit in the equality in (3.6) to find

$$\int_{\mathbb{R}^2} (|\nabla w_\delta|^2 + V_0|w_\delta|^2) = t_0^{-2} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(t_0 w_\delta) \right] f(t_0 w_\delta) t_0 w_\delta. \quad (3.7) \quad \boxed{\text{ESc2}}$$

Hence, from (3.5) and (3.7),

$$t_0^{-2} \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(t_0 w) \right] f(t_0 w) t_0 w - \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(w) \right] f(w) w = 0.$$

Thereby, by monotone assumption  $(f_5)$ , we derive that

$$t_0 = 1.$$

Since

$$\int_{\mathbb{R}^2} (V(\varepsilon_n x) - V_0)|v_n|^2 \rightarrow 0 \text{ and } \Phi_{V_0}(t_n v_n) \rightarrow \Phi_{V_0}(w_\delta),$$

the following inequality

$$c_{\varepsilon_n} \leq I_{\varepsilon_n}(t_n v_n) = \Phi_{V_0}(t_n v_n) + \frac{t_n^2}{2} \int_{\mathbb{R}^2} (V(\varepsilon_n x) - V_0)|v_n|^2,$$

gives

$$\limsup_{n \rightarrow +\infty} c_{\varepsilon_n} \leq \Phi_{V_0}(w_\delta) \leq m_{V_0} + \delta.$$

As  $\delta$  is arbitrary, we deduce that

$$\limsup_{n \rightarrow +\infty} c_{\varepsilon_n} \leq m_{V_0}.$$

As  $\varepsilon_n$  is also arbitrary, it follows that

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq m_{V_0}. \quad (3.8) \quad \boxed{\text{PASS01}}$$

On the other hand, we already know that

$$c_\varepsilon \geq m_{V_0}, \quad \forall \varepsilon > 0,$$

which implies

$$\liminf_{\varepsilon \rightarrow 0} c_\varepsilon \geq m_{V_0}. \quad (3.9) \quad \boxed{\text{PASS02}}$$

From (3.8) and (3.9) we get

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon \geq m_{V_0}.$$

and the proof follows by using Lemma 2.2. □

**PS** **Lemma 3.4.** *Suppose that assumptions  $(f_1) - (f_5)$ ,  $(V_1)$  and  $(V_2)$  hold. Let  $\{u_n\}$  be a  $(PS)_{c_\varepsilon}$  sequence with  $\varepsilon \in [0, \varepsilon_0)$ . Let  $u_\varepsilon$  be the weak limit of  $u_n$ , then  $\{u_n\}$  converges strongly to  $u_\varepsilon$  in  $E_\varepsilon$ , i.e.  $I_\varepsilon$  satisfies  $(PS)_{c_\varepsilon}$  condition for  $\varepsilon \in [0, \varepsilon_0)$ .*

*Proof.* First recall that

$$c_\varepsilon < \frac{4 - \mu}{8}, \quad \forall \varepsilon \in [0, \varepsilon_0) \quad (3.10)$$

$$m_{V_0} < m_{V_\infty}. \quad (3.11)$$

and there are positive constants  $a_1, a_2$  such that

$$a_1 < \|u_n\|_\varepsilon < a_2, \quad \forall n \in \mathbb{N} \quad (\text{for some subsequence}). \quad (3.12) \quad \boxed{\text{EST3}}$$

In the sequel, our first goal is to prove that  $u_\varepsilon \neq 0$ . To do that, we will argue by contradiction, assuming that  $u_\varepsilon = 0$ .

**Claim:** There exist  $\beta, \tilde{R} > 0$  and  $\{y_n\} \subset \mathbb{R}^2$  such that

$$\int_{B_{\tilde{R}}(y_n)} |u_n|^2 \geq \beta.$$

Otherwise, by applying a result due to Lions, we obtain

$$u_n \rightarrow 0 \quad \text{in} \quad L^q(\mathbb{R}^2) \quad \forall q \in (2, +\infty).$$

Following line by line the argument of Section 2, we have

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] F(u_n) \right| \rightarrow 0, \quad n \rightarrow \infty.$$

Since  $(u_n)$  be a  $(PS)_{c_\varepsilon}$  sequence with  $c_\varepsilon < \frac{4-\mu}{8}$ , we know that

$$\limsup_{n \rightarrow \infty} \|u_n\|_\varepsilon^2 = 2c_\varepsilon < \frac{4-\mu}{4}. \quad (3.13) \quad \boxed{\text{EST4}}$$

As in the proof of Theorem 1.3, we can conclude that

$$\left| \int_{\mathbb{R}^2} \left[ \frac{1}{|x|^\mu} * F(u_n) \right] f(u_n)u_n \right| \rightarrow 0, \quad n \rightarrow \infty.$$

This together with  $\langle I'_\varepsilon(u_n), u_n \rangle = o_n(1)$  implies that

$$\lim_{n \rightarrow +\infty} \|u_n\|_\varepsilon^2 = 0$$

which contradicts with (3.13), proving the claim.

Next, we fix  $t_n > 0$  such that  $t_n u_n \in \mathcal{N}_{V_\infty}$ . We claim that  $\{t_n\}$  is bounded. In fact, setting  $v_n = u_n(x + y_n)$ , by Claim 1, we may assume that, up to a subsequence,  $v_n \rightharpoonup v$  in  $E_\varepsilon$ . Moreover, using the fact that  $u_n \geq 0$  for all  $n \in \mathbb{N}$ , there exists  $a_3 > 0$  and a subset  $\Omega \subset \mathbb{R}^2$  with positive measure such that  $v(x) > a_3$  for all  $x \in \Omega$ . We have

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{F(t_n u_n(y)) f(t_n u_n(x)) t_n u_n(x)}{t_n^2 |x - y|^\mu} \right)$$

and so,

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{F(t_n v_n(y)) f(t_n v_n(x)) t_n v_n(x)}{t_n^2 |x - y|^\mu} \right)$$

from which

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) \geq \int_{\Omega} \int_{\Omega} \left( \frac{F(t_n v_n(y)) f(t_n v_n(x)) t_n v_n(x)}{t_n^2 |x - y|^\mu} \right)$$

Since

$$\liminf_{n \rightarrow \infty} \frac{F(t_n v_n(y)) f(t_n v_n(x)) t_n v_n(x)}{t_n^2 |x - y|^\mu} = +\infty \quad \text{a.e.}$$

Fatou's lemma gives

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) = +\infty,$$

which is a contradiction since  $\{u_n\}$  is bounded in  $E_\varepsilon$ . Thus, without loss of generality, we may assume

$$\lim_{n \rightarrow +\infty} t_n = t_0 > 0.$$

In what follows, we divide the remaining part of the proof into three steps.

**Step 1.** The number  $t_0$  is less or equal to 1.

In fact, suppose by contradiction that the above claim does not hold. Then, there exist  $\delta > 0$  and a subsequence of  $(t_n)$ , still denoted by itself, such that

$$t_n \geq 1 + \delta \quad \text{for all } n \in \mathbb{N}.$$



Since  $\langle I'_\varepsilon(u_n), u_n \rangle = o_n(1)$  and  $(t_n u_n) \subset \mathcal{N}_{V_\infty}$ , we have

$$\int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(\varepsilon x)|u_n|^2) = \mathfrak{F}'(u_n)[u_n] + o_n(1)$$

and

$$t_n^2 \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V_\infty |u_n|^2) = \mathfrak{F}'(t_n u_n)[t_n u_n].$$

Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^2} (V_\infty - V(\varepsilon x))|u_n|^2 + o_n(1) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{F(t_n u_n(y))f(t_n u_n(x))t_n u_n(x)}{t_n^2 |x - y|^\mu} - \frac{F(u_n(y))f(u_n(x))u_n(x)}{|x - y|^\mu} \right). \end{aligned}$$

Given  $\zeta > 0$ , from assumptions  $(V_1)$  and  $(V_2)$ , there exists  $R = R(\zeta) > 0$  such that

$$V(\varepsilon x) \geq V_\infty - \zeta, \text{ for any } |x| \geq R. \quad (3.14) \quad \boxed{\text{V1}}$$

Using the fact that  $u_n \rightarrow 0$  in  $L^2(B_R(0))$ , we conclude that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \frac{F(t_n u_n(y))f(t_n u_n(x))t_n u_n(x)}{t_n^2 |x - y|^\mu} - \frac{F(u_n(y))f(u_n(x))u_n(x)}{|x - y|^\mu} \right) \leq \zeta C + o_n(1),$$

where  $C = \sup_{n \in \mathbb{N}} |u_n|_2^2$ . Using the sequence  $v_n = u_n(x + y_n)$  again, we find the inequality

$$\begin{aligned} 0 &< \int_{\Omega} \int_{\Omega} \frac{|v_n(y)||v_n(x)|}{|x - y|^\mu} \left[ \frac{F((1 + \delta)v_n(y))f((1 + \delta)v_n(x))(1 + \delta)v_n(x)}{(1 + \delta)|v_n(y)|(1 + \delta)|v_n(x)|} \right. \\ &\quad \left. - \frac{F(v_n(y))f(v_n(x))v_n(x)}{|v_n(y)||v_n(x)|} \right] \\ &= \int_{\Omega} \int_{\Omega} \left[ \frac{F((1 + \delta)v_n(y))f((1 + \delta)v_n(x))(1 + \delta)v_n(x)}{(1 + \delta)^2 |x - y|^\mu} - \frac{F(v_n(y))f(v_n(x))v_n(x)}{|x - y|^\mu} \right] \\ &\leq \zeta C + o_n(1) \end{aligned}$$

Letting  $n \rightarrow \infty$  in the last inequality and applying Fatou's lemma, it follows that

$$0 < \int_{\Omega} \int_{\Omega} \frac{F((1 + \delta)v(y))f((1 + \delta)v(x))(1 + \delta)v(x)}{(1 + \delta)^2 |x - y|^\mu} - \frac{F(v(y))f(v(x))v(x)}{|x - y|^\mu} \leq \zeta C$$

which is absurd, since the arbitrariness of  $\zeta$ .

**Step 2.**  $t_0 = 1$ .

In this case, we begin with recalling that  $m_{V_\infty} \leq \Phi_{V_\infty}(t_n u_n)$ . Therefore,

$$c_\varepsilon + o_n(1) = I_\varepsilon(u_n) \geq I_\varepsilon(u_n) + m_{V_\infty} - \Phi_{V_\infty}(t_n u_n)$$

and from

$$\begin{aligned} I_\varepsilon(u_n) - \Phi_{V_\infty}(t_n u_n) &= \frac{(1 - t_n^2)}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^2} V(\varepsilon x)|u_n|^2 \\ &\quad - \frac{t_n^2}{2} \int_{\mathbb{R}^2} V_\infty |u_n|^2 + \mathfrak{F}(t_n u_n) - \mathfrak{F}(u_n), \end{aligned}$$

and the fact that  $\{u_n\}$  is bounded in  $E_\varepsilon$  as well as  $u_n \rightarrow 0$ , we derive from (3.14)

$$c_\varepsilon + o_n(1) \geq m_{V_\infty} - \zeta C + o_n(1),$$

and since  $\zeta$  is arbitrary we obtain

$$\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \geq m_{V_\infty},$$

which contradicts Lemma 3.3.

**Step 3.**  $t_0 < 1$ .

In this case, we may assume that  $t_n < 1$  for all  $n \in \mathbb{N}$ . Since  $m_{V_\infty} \leq \Phi_{V_\infty}(t_n u_n)$  and  $\langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \rangle = 0$ , we have

$$\begin{aligned} m_{V_\infty} &\leq \Phi_{V_\infty}(t_n u_n) - \frac{1}{2} \langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \rangle \\ &= \frac{1}{2} \mathfrak{F}'(t_n u_n)[t_n u_n] - \mathfrak{F}(t_n u_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(t_n u_n(y)) f(t_n u_n(x)) t_n u_n(x)}{|x - y|^\mu} - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(t_n u_n(y)) F(t_n u_n(x))}{|x - y|^\mu} \\ &< \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(u_n(y)) f(u_n(x)) u_n(x)}{|x - y|^\mu} - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(u_n(y)) F(u_n(x))}{|x - y|^\mu} \\ &= I_\varepsilon(u_n) - \frac{1}{2} \langle I'_\varepsilon(u_n), u_n \rangle \\ &= c_\varepsilon + o_n(1), \end{aligned}$$

which also yields a contradiction. From Steps 1, 2 and 3, we deduce that  $u_\varepsilon \neq 0$ . Hence, by Fatou's Lemma and using the characterization of  $c_\varepsilon$ , it follows that

$$\begin{aligned} c_\varepsilon &\leq I_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon) - \frac{1}{2} \langle I'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(u_\varepsilon(y)) [f(u_\varepsilon(x)) u_\varepsilon(x) - F(u_\varepsilon(x))]}{|x - y|^\mu} \\ &= \liminf_{n \rightarrow +\infty} \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{F(u_n(y)) [f(u_n(x)) u_n(x) - F(u_n(x))]}{|x - y|^\mu} \\ &\leq \limsup_{n \rightarrow +\infty} (I_\varepsilon(u_n) - \frac{1}{2} \langle I'_\varepsilon(u_n), u_n \rangle) = c_\varepsilon, \end{aligned}$$

thus

$$I_\varepsilon(u_\varepsilon) = c_\varepsilon.$$

Now, using the following inequalities

$$\begin{aligned} c_\varepsilon &= I_\varepsilon(u_\varepsilon) - \frac{1}{2K} \langle I'_\varepsilon(u_\varepsilon), u_\varepsilon \rangle \\ &\leq \liminf_{n \rightarrow +\infty} (I_\varepsilon(u_n) - \frac{1}{2K} \langle I'_\varepsilon(u_n), u_n \rangle) \\ &\leq \limsup_{n \rightarrow +\infty} (I_\varepsilon(u_n) - \frac{1}{2K} \langle I'_\varepsilon(u_n), u_n \rangle) \\ &= c_\varepsilon \end{aligned}$$

we actually have

$$u_n \rightarrow u_\varepsilon \quad \text{in} \quad E_\varepsilon,$$

showing that  $I_\varepsilon$  verifies the  $(PS)_{c_\varepsilon}$  condition.  $\square$

As an immediate consequence of Lemma 3.4, we have

**Existence** **Corollary 3.5.** *The minimax value  $c_\varepsilon$  is achieved if  $\varepsilon$  is small enough and hence problem  $(SNS^*)$  has a solution of least energy if  $\varepsilon$  is small enough.*

## 4 Concentration phenomena: proof of Theorem 1.4 completed

In this section our goal is to establish the concentration phenomenon for ground state solutions of the singularly perturbed equation  $(SNS^*)$ . For this purpose, the following technical lemma will play a fundamental role.

**BNT1** **Lemma 4.1.** *Suppose that assumptions  $(f_1)$  and  $(f_2)$  hold. If  $h \in H^1(\mathbb{R}^2)$ , then the function  $\frac{1}{|x|^\mu} * F(h)$  belongs to  $L^\infty(\mathbb{R}^2)$ .*

*Proof.* For  $\beta > 1$ , there exists  $C_0 > 0$  such that

$$F(s) \leq C_0 \left( |s|^{\frac{4-\mu}{2}} + |s| [e^{\beta 4\pi s^2} - 1] \right), \forall s \in \mathbb{R}.$$

Then,

$$\begin{aligned} \left| \frac{1}{|x|^\mu} * F(h) \right| &= \left| \int_{\mathbb{R}^2} \frac{F(h)}{|x-y|^\mu} \right| \\ &= \left| \int_{|x-y| \leq 1} \frac{F(h)}{|x-y|^\mu} \right| + C \left| \int_{|x-y| \geq 1} \frac{F(h)}{|x-y|^\mu} \right| \\ &\leq \int_{|x-y| \leq 1} \frac{|h|^{\frac{4-\mu}{2}} + |h| [e^{\beta 4\pi |h|^2} - 1]}{|x-y|^\mu} \\ &\quad + C \int_{|x-y| \geq 1} \left( \frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} + |h| [e^{\beta 4\pi |h|^2} - 1] \right). \end{aligned}$$

Since

$$\frac{1}{|y|^\mu} \in L^{\frac{2+\delta}{\mu}}(B_1^c(0)), \quad \forall \delta > 0,$$

take  $\delta \approx 0^+$  such that

$$q_{1,\delta} = \frac{(4-\mu)}{2} \frac{(2+\delta)}{(2+\delta)-\mu} > 2.$$

Using Hölder inequality, we get

$$\int_{|x-y| \geq 1} \frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} \leq C_0 \left( \int_{|x-y| \geq 1} |h|^{q_{1,\delta}} \right)^{\frac{(2+\delta)-\mu}{2+\delta}} = C_1.$$

On the other hand, by Lemma 1.2

$$e^{2\beta 4s\pi|h|^2} - 1 \in L^1(\mathbb{R}^2), \quad \forall s \geq 1,$$

Again by Hölder's inequality

$$\int_{|x-y|\geq 1} |h| [e^{\beta 4\pi |h|^2} - 1] \leq |h|_2 \int_{\mathbb{R}^2} \left( [e^{2\beta 4\pi \frac{|h|^2}{\|h\|_2^2}} - 1] \right)^{\frac{1}{2}} \leq C_2.$$

for some positive constant  $C_2$ .

Choosing  $t \in (\frac{2}{2-\mu}, +\infty)$ , we have that  $\frac{(4-\mu)t}{2} > 2$  and  $1 - \frac{t\mu}{t-1} > -1$ . Then, from Hölder's inequality

$$\begin{aligned} \int_{|x-y|\leq 1} \frac{|h|^{\frac{4-\mu}{2}}}{|x-y|^\mu} &\leq \left( \int_{|x-y|\leq 1} |h|^{\frac{(4-\mu)t}{2}} \right)^{\frac{1}{t}} \left( \int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \right)^{\frac{t-1}{t}} \\ &\leq C_2 \left( \int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \right)^{\frac{t-1}{t}} = C_3. \end{aligned}$$

Furthermore, using again Lemma 1.2, we get

$$\begin{aligned} \int_{|x-y|\leq 1} \frac{|h| [e^{\beta 4\pi |h|^2} - 1]}{|x-y|^\mu} &\leq \left( \int_{|x-y|\leq 1} |h| [e^{\beta 4\pi |h|^2} - 1]^t \right)^{\frac{1}{t}} \left( \int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \right)^{\frac{t-1}{t}} \\ &\leq \left( \int_{|x-y|\leq 1} |h|^{2t} \right)^{\frac{1}{2t}} \left( \int_{|x-y|\leq 1} [e^{2\beta 4\pi |h|^2} - 1] \right)^{\frac{1}{2t}} \left( \int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \right)^{\frac{t-1}{t}} \\ &\leq C_4. \end{aligned}$$

The lemma thus follows from the above estimates.  $\square$

**Seq**

**Proposition 4.2.** *Let  $\varepsilon_n \rightarrow 0$  and  $\{u_n\}$  be the sequence of solutions obtained in Corollary 3.5. Then, there exists a sequence  $\{y_n\} \subset \mathbb{R}^2$ , such that  $v_n = u_n(x + y_n)$  has a convergent subsequence in  $E$ . Moreover, up to a subsequence,  $y_n \rightarrow y \in M$ .*

*Proof.* Let  $\{u_n\}$  be the sequence of solutions obtained in Corollary 3.5, it is easy to see  $c_{\varepsilon_n} = I_{\varepsilon_n}(u_n) \rightarrow m_{V_0}$ ,  $\{u_n\}$  is bounded in  $E$  and

$$0 < m_{V_0} = \limsup_{n \rightarrow \infty} c_{\varepsilon_n} < \frac{(4-\mu)}{8}.$$

By following the argument in the proof of Theorem 1.3 in Section 2, there exist  $r, \delta > 0$  and  $\tilde{y}_n \in \mathbb{R}^2$  such that

$$\liminf_{n \rightarrow \infty} \int_{B_r(\tilde{y}_n)} |u_n|^2 \geq \delta. \quad (4.1) \quad \boxed{\text{B1'}}$$

Setting  $v_n(x) = u_n(x + \tilde{y}_n)$ , up to a subsequence, if necessary, we may assume  $v_n \rightharpoonup v \neq 0$  in  $E$ . Let  $t_n > 0$  be such that  $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$ . Then,

$$m_{V_0} \leq \Phi_{V_0}(\tilde{v}_n) = \Phi_{V_0}(t_n u_n) \leq I_\varepsilon(t_n u_n) \leq I_\varepsilon(u_n) \rightarrow m_{V_0}$$

and so,

$$\Phi_{V_0}(\tilde{v}_n) \rightarrow m_{V_0} \text{ and } (\tilde{v}_n) \subset \mathcal{N}_{V_0}.$$

Then the sequence  $\{\tilde{v}_n\}$  is a minimizing sequence, and by the Ekeland Variational Principle [23], we may also assume it is a bounded  $(PS)$  sequence at  $m_{V_0}$ . Thus, for some subsequence,  $\tilde{v}_n \rightharpoonup \tilde{v}$  weakly in  $E$  with  $\tilde{v} \neq 0$  and  $\Phi'_{V_0}(\tilde{v}) = 0$ . Repeating the same arguments used in the proof of Lemma 3.4, we have that  $\tilde{v}_n \rightarrow \tilde{v}$  in  $E$ . Since  $(t_n)$  is bounded, we can assume that for some subsequence  $t_n \rightarrow t_0 > 0$ , and so  $v_n \rightarrow v$  in  $E$ .

Next we will show that  $\{y_n\} = \{\varepsilon_n \tilde{y}_n\}$  has a subsequence satisfying  $y_n \rightarrow y \in M$ . We begin with proving that  $\{y_n\}$  is bounded in  $\mathbb{R}^2$ . Indeed, if not there would exist a subsequence, which we still denote by  $\{y_n\}$ , such that  $|y_n| \rightarrow \infty$ . Since  $\tilde{v}_n \rightarrow \tilde{v}$  in  $E$  and  $V_0 < V_\infty$ , we have

$$\begin{aligned} m_{V_0} &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 + \frac{1}{2} \int_{\mathbb{R}^2} V_0 |\tilde{v}|^2 - \mathfrak{F}(\tilde{v}) \\ &< \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 + \frac{1}{2} \int_{\mathbb{R}^2} V_\infty |\tilde{v}|^2 - \mathfrak{F}(\tilde{v}) \\ &\leq \liminf_{n \rightarrow \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \tilde{v}_n|^2 + \frac{1}{2} \int_{\mathbb{R}^2} V(\varepsilon_n x + y_n) |\tilde{v}_n|^2 - \mathfrak{F}(\tilde{v}_n) \right] \\ &= \liminf_{n \rightarrow \infty} \left[ \frac{t_n^2}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{t_n^2}{2} \int_{\mathbb{R}^2} V(\varepsilon_n x) |u_n|^2 - \mathfrak{F}(t_n^2 u_n) \right] \\ &= \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(t_n u_n) \\ &\leq \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n) \\ &= m_{V_0} \end{aligned}$$

hence the absurd which shows that  $\{y_n\}$  stays bounded and up to a subsequence,  $y_n \rightarrow y \in \mathbb{R}^2$ . Then, necessarily  $y \in M$  otherwise we would get again a contradiction as above.  $\square$

Let  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $u_n$  be the ground state solution of

$$-\Delta u + V(\varepsilon_n x)u = \left[ \frac{1}{|x|^\mu} * F(u) \right] f(u) \quad \text{in } \mathbb{R}^2.$$

From Lemma 3.3 we know

$$I_{\varepsilon_n}(u_n) \rightarrow m_{V_0}.$$

Then, there exists a sequence  $\tilde{y}_n \in \mathbb{R}^2$ , such that  $v_n = u_n(x + \tilde{y}_n)$  is a solution of

$$-\Delta v_n + V_n(x)v_n = \left[ \frac{1}{|x|^\mu} * F(v_n) \right] f(v_n), \quad \text{in } \mathbb{R}^2,$$

where  $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ . Moreover,  $(v_n)$  has a convergent subsequence in  $E$  and  $y_n \rightarrow y \in M$ , up to a subsequence, where  $y_n = \varepsilon_n \tilde{y}_n$ . Hence, there exists  $h \in H^1(\mathbb{R}^2)$  such that

$$|v_n(x)| \leq h(x) \quad \text{a.e in } \mathbb{R}^2 \quad \forall n \in \mathbb{N}. \quad (4.2) \quad \square$$

**Lemma 4.3.** *Suppose that conditions  $(f_1) - (f_5)$ ,  $(V_1)$  and  $(V_2)$  hold. Then there exists  $C > 0$  such that  $\|v_n\|_{L^\infty(\mathbb{R}^2)} \leq C$  for all  $n \in \mathbb{N}$ . Furthermore*

$$\lim_{|x| \rightarrow \infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

*Proof.* Let us first show that the sequence

$$W_n(x) := \left[ \frac{1}{|x|^\mu} * F(v_n) \right],$$

stays bounded in  $L^\infty(\mathbb{R}^2)$ . Indeed, as  $F$  is an increasing function, by (4.2) we know that

$$0 \leq W_n(x) := \left[ \frac{1}{|x|^\mu} * F(v_n) \right] \leq \left[ \frac{1}{|x|^\mu} * F(h) \right]$$

Hence claim will hold provided the function

$$W(x) = \left[ \frac{1}{|x|^\mu} * F(h) \right]$$

belongs to  $L^\infty(\mathbb{R}^2)$  and this is an immediate consequence of Lemma 4.1.

For any  $R > 0$ ,  $0 < r \leq \frac{R}{2}$ , let  $\eta \in C^\infty(\mathbb{R}^2)$ ,  $0 \leq \eta \leq 1$  with  $\eta(x) = 1$  if  $|x| \geq R$  and  $\eta(x) = 0$  if  $|x| \leq R - r$  and  $|\nabla \eta| \leq \frac{2}{r}$ . For  $L > 0$ , let

$$v_{L,n} = \begin{cases} v_n(x), & v(x) \leq L \\ L, & v_n(x) \geq L, \end{cases}$$

and

$$z_{L,n} = \eta^2 v_{L,n}^{2(\gamma-1)} v_n \quad \text{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\gamma-1}$$

with  $\gamma > 1$  to be determined later. Taking  $z_{L,n}$  as a test function, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)} |\nabla v_n|^2 + \int_{\mathbb{R}^2} \tilde{V}_{\varepsilon_n}(x) |v_n|^2 \eta^2 v_{L,n}^{2(\gamma-1)} \\ &= -2(\gamma-1) \int_{\mathbb{R}^2} v_n v_{L,n}^{2\gamma-3} \eta^2 \nabla v_n \nabla v_{L,n} + \int_{\mathbb{R}^2} W_n(x) f(v_n) \eta^2 v_n v_{L,n}^{2(\gamma-1)} \\ & \quad - 2 \int_{\mathbb{R}^2} \eta v_{L,n}^{2(\gamma-1)} v_n \nabla v_n \nabla \eta. \end{aligned} \tag{4.3} \quad \boxed{\text{E1}}$$

Using Lemma 1.2, for all  $\beta, s > 1$ , we know that

$$\int_{\mathbb{R}^2} [e^{\beta 4\pi v_n^2} - 1]^s \leq \int_{\mathbb{R}^2} [e^{\beta 4\pi |h|^2} - 1]^s = C < \infty \quad \forall n \in \mathbb{N}. \tag{4.4} \quad \boxed{\text{E2}}$$

Let  $t = \sqrt{s}$ ,  $p > \frac{2t}{t-1} > 2$  and  $\gamma = \frac{p(t-1)}{2t}$ , for any  $\delta > 0$ , there exists  $C(\delta, p, \beta) > 0$  such that

$$F(u) \leq \delta u^2 + C(\delta, p, \beta) u^{p-1} [e^{\beta 4\pi |u|^2} - 1], \quad \forall u \in \mathbb{R}.$$

Thus for  $\delta$  sufficiently small, as  $(W_n)$  is bounded in  $L^\infty(\mathbb{R}^2)$ , gathering (4.3) and Young's inequality, we get

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)} |\nabla v_n|^2 + V_0 \int_{\mathbb{R}^2} |v_n|^2 \eta^2 v_{L,n}^{2(\gamma-1)} \\ & \leq C \int_{\mathbb{R}^2} v_n^p \eta^2 v_{L,n}^{2(\gamma-1)} [e^{\beta 4\pi |h|^2} - 1] + C \int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)} |\nabla \eta|^2. \end{aligned} \tag{4.5} \quad \boxed{\text{E3}}$$

Using this fact, from [4] we have

$$|w_{L,n}|_p^2 \leq C\gamma^2 \left( C' + \left[ \int_{|x| \geq R-r} v_n^{(p-2)t} [e^{\beta 4\pi|h|^2} - 1]^t \right]^{\frac{1}{t}} \right) \left[ \int_{|x| \geq R-r} v_n^{\frac{2\gamma t}{t-1}} \right]^{\frac{t-1}{t}}.$$

By (4.4) and Hölder's inequality, we know

$$|w_{L,n}|_p^2 \leq C\gamma^2 \left[ \int_{|x| \geq R-r} v_n^{\frac{2\gamma t}{t-1}} \right]^{\frac{t-1}{t}}.$$

Now, following the same iteration arguments explored in [4], we find

$$|v_n|_{L^\infty(|x| \geq R)} \leq C|v_n|_{p(|x| \geq R/2)}. \quad (4.6) \quad \boxed{\text{BD1}}$$

For  $x_0 \in B_R$ , we can use the same argument taking  $\eta \in C_0^\infty(\mathbb{R}^2, [0, 1])$  with  $\eta(x) = 1$  if  $|x - x_0| \leq \rho'$  and  $\eta(x) = 0$  if  $|x - x_0| > 2\rho'$  and  $|\nabla \eta| \leq \frac{2}{\rho'}$ , to prove that

$$|v_n|_{L^\infty(|x-x_0| \leq \rho')} \leq C|v_n|_{p(|x| \leq 2\rho')}. \quad (4.7) \quad \boxed{\text{BD2}}$$

With (4.6) and (4.7), by a standard covering argument it follows that

$$|v_n|_\infty < C$$

for some positive constant  $C$ . Then, using again the convergence of  $(v_n)$  to  $v$  in  $E$  in the right side of (4.6), for each  $\delta > 0$  fixed, there exists  $R > 0$  such that  $|v_n|_{L^\infty(|x| \geq R)} < \delta, \forall n \in \mathbb{N}$ . Thus,

$$\lim_{|x| \rightarrow \infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N},$$

and the proof is complete.  $\square$

The last lemma establishes an estimate from below in terms of the  $L^\infty$ -norm of  $\{v_n\}$ .

$\boxed{\text{MP}}$  **Lemma 4.4.** *There exists  $\delta_0 > 0$  such that  $|v_n|_\infty \geq \delta_0$  for all  $n \in \mathbb{N}$ .*

*Proof.* Recall that,

$$\delta \leq \int_{B_r(\tilde{y}_n)} |u_n|^2,$$

then

$$\delta \leq \int_{B_r(0)} |v_n|^2 \leq |B_r| |v_n|_\infty^2,$$

from where it follows

$$|v_n|_\infty \geq \delta_0,$$

showing the lemma.  $\square$

**Concentration around maxima.** Let  $b_n$  denote a maximum point of  $v_n$ , we know it is a bounded sequence in  $\mathbb{R}^2$ . Thus, there is  $R > 0$  such that  $b_n \in B_R(0)$ . Thus the global maximum of  $u_{\varepsilon_n}$  is attained at  $z_n = b_n + \tilde{y}_n$  and

$$\varepsilon_n z_n = \varepsilon_n b_n + \varepsilon_n \tilde{y}_n = \varepsilon_n b_n + y_n.$$

From the boundedness of  $\{b_n\}$  we have

$$\lim_{n \rightarrow \infty} z_n = y,$$

which together with the continuity of  $V$  yields

$$\lim_{n \rightarrow \infty} V(\varepsilon_n z_n) = V_0.$$

If  $u_\varepsilon$  is a positive solution of  $(SNS^*)$  the function  $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$  is a positive solution of (1.8). Thus, the maxima points  $\eta_\varepsilon$  and  $z_\varepsilon$  of respectively  $w_\varepsilon$  and  $u_\varepsilon$ , satisfy the equality  $\eta_\varepsilon = \varepsilon z_\varepsilon$  and in turn

$$\lim_{\varepsilon \rightarrow 0} V(\eta_\varepsilon) = V_0.$$

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